

Solution of Hypergraph Turan problem

Vladimir Blinovsky*

Instituto de Matematica e Estatística, USP,
Rua do Matao 1010, 05508-090, São Paulo, Brazil
Institute for Information Transmission Problems,
B. Karetnyi 19, Moscow, Russia,
vblinovs@yandex.ru

Abstract

Using original *Symmetrical Smoothing Method* we solve hypergraph $(3, k)$ -Turan problem

Let X be a finite set $|X| = n$. Define $\binom{X}{k}$ to be the family of k -element subsets of X . We say that family (hypergraph) $\mathcal{A} \subset \binom{X}{3}$ satisfies $(3, k)$ -Turan property if for the arbitrary $A \in \binom{[n]}{k}$ it follows that

$$|B \in \mathcal{A} : B \in A| < \binom{k}{3}.$$

There is a number of sites and conferences devoted to this problem (see [3], [4], [5]). We investigate the following

Problem. For given $n > k > 3$ find the maximal (at least one) family which satisfies $(3, k)$ -Turan property.

This is famous Turan problem. Actually Turan problem in general case (not only 3-hypergraph but m -hypergraph) is the key problem in Erdos's extremal combinatorics. And the case $m = 3$ is the first nontrivial case of great importance. For the surveys and references see [1], [2].

*The author was supported by the São Paulo Research Foundation (FAPESP), Project no 2014/23368-6 and NUMEC/USP (Project MaCLinC/USP).

Next we assume that $(k-1)|n$. For other cases it is necessary to find proper explicit symmetrical constructions for the optimal Turan hypergraph. I am lazy to do this and it is not needed for asymptotic Turan problem (see [7], [6]).

To solve this problem we make some preliminary preparations.

We use the natural bijection between $2^{[n]}$ and $\{0,1\}^n$ and don't make difference between these two sets.

Define

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\xi^2/2} d\xi,$$

$$\begin{aligned} N(\{\sigma\}) &= \sum_{x \in \binom{[n]}{3}} \varphi \left(\left(\left(\sum_{i=1}^M \varphi((x, \beta_i) - 3)/\sigma \right) - 1/2 \right) / \sigma \right), \\ R(\{\sigma\}) &= \binom{n}{k} - \sum_K \sum_{x \in K} \varphi \left(\left(\left(\varphi \left(\left(\sum_{i=1}^M \varphi((x, \beta_i) - 3)/\sigma \right) - 1/2 \right) / \sigma \right) - (1 - \epsilon) \right) / \sigma \right). \end{aligned}$$

Here K 's are complete 3-hypergraphs on vertices $K \subset [n]$, $|K| = k$. It can be easily seen that N is convex and R is concave function of $\{\beta_i\}$. We assume that only three coordinates are nonzero and these nonzero coordinates are equal to 1.

We have for $N \rightarrow M/2$ as $\sigma \rightarrow 0$ and if hypergraph has $(3, k)$ -Turan property, then $R = o(1)$ as $\sigma \rightarrow 0$.

Necessary and sufficient condition for minimization of M is the Kuhn- Tucker conditions:

$$\begin{aligned} N'_{\beta_{i,j}} &= \lambda R'_{\beta_{i,j}}, \\ R &= o(1), \sigma \rightarrow \infty. \end{aligned} \tag{1}$$

Consider the following well known construction of $(3, k)$ -Turan hypergraph. We describe the complement hypergraph $\bar{T} = \binom{[n]}{3} \setminus T$. We divide set of vertices $[n]$ into $k-1$ equal parts B_1, \dots, B_{k-1} of size $n/(k-1)$. Hypergraph \bar{T} consists of edges in each part B_i and all edges such that each of them has two vertices in B_i and one vertex in $B_{i+1 \bmod k-1}$.

It is easy to see, that T satisfies $(3, k)$ -Turan property.

Note that we find solution of (1) not among all vertices β_i but with specific condition that only three coordinates $\beta_{i,1}, \beta_{i,2}, \beta_{i,3}$ are nonzero and $\beta_{i,1} + \beta_{i,2} + \beta_{i,3} = 3$.

Assume that $\beta_{i,1} = \beta_{i,2} = \beta_{i,3} = 1$. It is easy to see that

$$N'_{\beta_{i,1}} = R'_{\beta_{i,1}} = 0$$

if $(\beta_{i_1,1}, \beta_{i_2,2})$ are on the positions of x which belongs the same cell \bar{T}_r and

$$\begin{aligned} N'_{\beta_{i_1,1}} &= N'_{\beta_{i_2,2}}, \\ R'_{\beta_{i_1,1}} &= R'_{\beta_{i_2,2}} \end{aligned}$$

if $(\beta_{i_1,1}, \beta_{i_2,2})$ are components of consecutive (cyclically) sells. From here it follows that there exists unique λ such that equations (1) are satisfied and hence construction of T is optimal.

Easy consequence of optimality of T is the validity of the following famous Turan

Conjecture 1

$$\lim_{n \rightarrow \infty} \frac{|T|}{\binom{n}{3}} = 1 - \left(\frac{2}{k-1} \right)^2.$$

At last note that original Symmetrical Smoothing Method allows to prove optimality of the solution of extremal problems for (binary and not only) sequences in many cases when sufficiently symmetric constructions which are optimal are suggested. We are going to show how to solve many such problems in forthcoming papers.

References

- [1] Dhruv Mubayi, Oleg Pikhurko, Benny Sudakov, Hypergraph Turan Problem: Some Open Questions, <http://homepages.warwick.ac.uk/~maskat/Papers/TuranQuestions.pdf>,
- [2] Peter Keevash, Hypergraph Turan Problems, <http://people.maths.ox.ac.uk/keevash/papers/turan-survey.pdf>
- [3] <http://aimpl.org/hypergraphturan/archives/0.31/1/>
- [4] <http://www.openproblemgarden.org/category/turan-paul>
- [5] <https://www.mfo.de/occasion/1215b/www-view>
- [6] More Constructions for Turans (3, 4)-Conjecture Andrew Frohmader September 2, 2008, <http://www.math.cornell.edu/~froh/turanconj.pdf>
- [7] A. Kostochka, A class of constructions for Turans (3,4) problem, Combinatorica 2 (1982), 187192.